VOLUME ENTROPY OF HILBERT METRICS AND LENGTH SPECTRUM OF HITCHIN REPRESENTATIONS INTO $\mathrm{PSL}(3,\mathbb{R})$

NICOLAS THOLOZAN

Abstract. This article studies the geometry of proper open convex domains in the projective space $\mathbb{R}\mathbf{P}^n$. These domains carry several projective invariant distances, among which the Hilbert distance d^H and the Blaschke distance d^B. We prove a thin inequality between those distances: for any two points x and y in such a domain,

$$d^B(x,y) < d^H(x,y) + 1 \ .$$

We then give two interesting consequences. The first one answers to a conjecture of Colbois and Verovic on the volume entropy of Hilbert geometries: for any proper open convex domain in $\mathbb{R}\mathbf{P}^n$, the volume of a ball of radius R grows at most like $e^{(n-1)R}$. The second consequence is the following fact: for any Hitchin representation ρ of a surface group Γ into $\mathrm{PSL}(3,\mathbb{R})$, there exists a Fuchsian representation $j : \Gamma \to \mathrm{PSL}(2,\mathbb{R})$ such that the length spectrum of j is uniformly smaller than that of ρ. This answers positively to a conjecture of Lee and Zhang in the three-dimensional case.

Contents

Introduction

An open domain in $\mathbb{R}\mathbf{P}^n$ is *convex* if its intersection with any projective line is connected. It is called *proper* if its closure does not contain a projective line.

The geometry of proper open convex domains in $\mathbb{R}\mathbf{P}^n$ has been extensively studied since Hilbert introduced them as examples of metric spaces "whose geodesics are straight lines" (see [15]). More precisely, Hilbert equipped any proper open convex domain with a natural Finsler metric for which projective segments are geodesics. Moreover, this metric is a projective invariant and therefore any projective transformation preserving such a domain is an isometry with respect to the Hilbert metric. When the convex domain is a ball, we recover the Klein model of the hyperbolic space.

The Hilbert metric seems to be the most natural metric on a proper open convex domain in $\mathbb{R}\mathbf{P}^n$, but it is not so easy to deal with. Indeed, it is (almost) never Riemannian and in many interesting cases it is not $\mathcal{C}^2$. Another choice of a projectively invariant metric on such a domain is the *Blaschke metric* (also known as the *affine metric*) that arises in the theory of affine spheres developed by Blaschke, Calabi, Cheng and Yau. The definition of the Blaschke metric relies on a deep theorem in analysis and may seem difficult to handle at first glance. The counterpart to this is that it is a smooth Riemannian metric with nice curvature properties (see Theorem 0.5).

One may hope in general that the Blaschke metric is "close enough" to the Hilbert metric so that all the good analytic properties of the Blaschke metric give similar properties for the Hilbert metric.

0.1. A comparison lemma.

Let us fix a proper open convex domain Ω in $\mathbb{R}\mathbf{P}^n$. Denote by h_Ω^H and h_Ω^B its Hilbert and Blaschke metrics, respectively, and by d_Ω^H and d_Ω^B their associated distances. (Very often, we will omit to index these objects by Ω.) In a recent paper, Benoist and Hulin proved that Blaschke and Hilbert metrics are uniformly comparable.

Theorem 0.1 (Benoist–Hulin, [3])**.** *There exists a positive constant C_n such that, for any proper open convex domain $\Omega \subset \mathbb{R}\mathbf{P}^n$, one has*

$$\frac{1}{C_n} h_\Omega^H \leq h_\Omega^B \leq C_n h_\Omega^H \ .$$

The central result of this paper is a refinement of the right-hand inequality:

Main Lemma.
For any $x, y \in \Omega$, we have

$$d^B(x,y) < d^H(x,y) + 1 \ .$$

Remark 0.2. Clearly, this lemma only refines Benoist–Hulin's theorem when $d^H(x,y)$ is big enough. In Subsection 2.1, we prove a refined version of the Main Lemma where the additive constant is related to the multiplicative constant of Theorem 0.1.

Remark 0.3. One may hope for a stronger inequality, namely $d^B \leq d^H$. However, computing both metrics when Ω is a square in $\mathbb{R}\mathbf{P}^2$ shows that this stronger inequality does not always hold.

We will now give two important consequences of the Main Lemma.

0.2. **Volume entropy of Hilbert metrics.** Given a proper open convex domain $\Omega \subset \mathbb{R}\mathbf{P}^n$ there is no standard way to associate a volume form to its Hilbert metric h^H, but there is a natural class of volume forms.

We call a volume form *vol* on Ω *uniform* if there exists a constant $K \geq 1$ such that for any point $x \in \Omega$, one has

$$\frac{1}{K} \leq vol_x\left(\{u \in T_x\Omega \mid h^H(u) \leq 1\}\right) \leq K \ .$$

Note that, according to Theorem 0.1, an example of such a volume form is the one canonically associated with the Blaschke metric on Ω.

Denote by $B^H(x, R)$ the ball of radius R about x with respect to the Hilbert metric on Ω.

Definition 0.4. The *volume entropy* of the Hilbert metric h^H on Ω is defined by

$$\mathcal{H}(h^H) = \limsup_{R \to +\infty} \frac{1}{R} \log \mathbf{Vol}\left(B^H(o, R)\right) \ ,$$

where o is some base point in Ω and the volume **Vol** is computed with respect to a uniform volume form *vol* on Ω.

It is not difficult to see that this definition does not depend on the volume form *vol*, nor on the base point o. One can define in the same way the volume entropy of the Blaschke metric h^B by replacing $B^H(o, R)$ by $B^B(o, R)$, the ball of radius R about o with respect to h^B. The volume entropy of the Hilbert metric is sometimes called the *volume entropy of* Ω, but for our purpose it is better to distinguish between the volume entropies of h^B and h^H.

It is a well-known conjecture in Hilbert geometry that the volume entropy of the Hilbert metric of a proper open convex domain in $\mathbb{R}\mathbf{P}^n$ is bounded from above by $n - 1$. This conjecture seems to date back from a work of Colbois and Verovic where it is proved that, if the boundary of Ω is sufficiently regular, then the volume entropy of Ω is actually equal to $n - 1$ (see [10]). This was later refined by Berck, Bernig and Vernicos in [5], where they also proved the conjecture in dimension 2. Vernicos then recently proved in [24] the conjecture in dimension 3.

In another direction, Crampon proved in [11] the conjecture in any dimension when assuming Ω is *divisible* and *hyperbolic* (*i.e.*, preserved by a discrete Gromov-hyperbolic group Γ acting cocompactly). In that case, the volume entropy of the Hilbert metric is actually the topological entropy of the geodesic flow on Ω/Γ.

Here we prove the conjecture in full generality:

Theorem A.
Let Ω be a proper open convex domain in $\mathbb{R}\mathbf{P}^n$. Then the volume entropy of the Hilbert metric on Ω satisfies

$$\mathcal{H}\left(h^H\right) \leq n - 1 \ .$$

As we said, the main ingredient in the proof of this theorem is the Main Lemma. Indeed, this lemma implies the inequality

$$\mathcal{H}\left(h^H\right) \leq \mathcal{H}\left(h^B\right)$$

(see Lemma 2.7), and therefore the only thing to be proved is that $\mathcal{H}(h^B) \leq n-1$. This is a consequence of a famous theorem of Bishop for the volume entropy of Riemannian metrics (Theorem 2.8) together with the following theorem of Calabi giving a lower bound for the Ricci curvature of the Blaschke metric:

Theorem 0.5 (Calabi, [7])**.** *The Ricci curvature of the Blaschke metric h^B on Ω satisfies*

$$-(n-1)h^B \leq \mathrm{Ricci}(h^B) \leq 0 \ .$$

In the case where a Gromov-hyperbolic group Γ acts properly discontinuously and cocompactly on Ω, Crampon proved that the entropy equals $n-1$ if and only if Ω is an ellipsoid and Γ a hyperbolic lattice. Using the present work, Barthelmé, Marquis and Zimmer recently improved this result by removing the *a priori* condition that Γ is Gromov-hyperbolic and by assuming that it only acts with finite covolume (see [2]).

In general, the volume entropy equals $n-1$ as soon as the boundary of Ω is of class $\mathcal{C}^{1,1}$ (see [5]). One may expect that, converserly, the volume entropy being equal to $n-1$ implies a certain regularity of the boundary of Ω.

0.3. **Length spectrum of Hitchin representations into** $\mathrm{PSL}(3,\mathbb{R})$**.** Let g be an isometry of some metric space (X,d). We define the *translation length* of g as the number

$$l(g) = \lim_{n\to+\infty} \frac{1}{n} d(x, g^n \cdot x) \ ,$$

where x is a point in X (it does not depend on the choice of x).

Definition 0.6. The *length spectrum* of a representation $\rho : \Gamma \to \mathrm{Isom}(X,d)$ is the function L_ρ that associates to (the conjugacy class of) an element $\gamma \in \Gamma$ the translation length of $\rho(\gamma)$.

We now assume that Γ is the fundamental group of a closed connected oriented surface S of genus greater than 1. A representation $\rho : \Gamma \to \mathrm{PSL}(3,\mathbb{R})$ is called a *Hitchin representation* if ρ is injective and $\rho(\Gamma)$ divides a (necessarily unique) open convex domain Ω_ρ in $\mathbb{R}\mathbf{P}^2$, which means that $\rho(\Gamma)$ acts properly discontinuously and cocompactly on Ω_ρ. (The terminology "Hitchin representation" will be explained in the next subsection.)

One can therefore define the length spectrum of a Hitchin representation $\rho : \Gamma \to \mathrm{PSL}(3,\mathbb{R})$ by considering ρ as a representation of Γ into the isometry group of (Ω_ρ, h^H).

Denote by $\mathbb{H}^2$ the hyperbolic plane. Recall that a representation $j : \Gamma \to \mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{Isom}^+(\mathbb{H}^2)$ is *Fuchsian* if it is injective and acts properly discontinuously on $\mathbb{H}^2$. The isomorphism $\mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{SO}_0(2,1)$ identifies Fuchsian representations with those Hitchin representations into $\mathrm{PSL}(3,\mathbb{R})$ that divide a disc in $\mathbb{R}\mathbf{P}^2$.

Motivated by questions arising in anti-de Sitter geometry, the author recently proved with Deroin a strong "domination" result for certain representations of a surface group.

Theorem 0.7 (Deroin–Tholozan, [13])**.** *Let ρ be a representation of Γ into the isometry group of a complete, simply connected Riemannian manifold of sectional curvature bounded from above by -1. Then there exists a Fuchsian representation $j : \Gamma \to \mathrm{PSL}(2,\mathbb{R})$ such that*

$$L_j \geq L_\rho \ .$$

Moreover, the inequality in this theorem can be made strict unless ρ itself is "Fuchsian" in some very rigid sense. This applies mostly to representations into Lie groups of rank 1 (seen as isometry groups of their symmetric spaces). In the case of $\mathrm{PSL}(2,\mathbb{C})$, it gives a new proof of Bowen's famous rigidity theorem for the entropy of quasi-Fuchsian representations (see [6]).

The Main Lemma allows us to prove a similar result (but with reverse inequality) for Hitchin representations into $\mathrm{PSL}(3,\mathbb{R})$.

Theorem B.
Let ρ be a Hitchin representation of Γ into $\mathrm{PSL}(3,\mathbb{R})$. Then either ρ is Fuchsian or there exist a constant $K > 1$ and a Fuchsian representation $j : \Gamma \to \mathrm{PSL}(2,\mathbb{R})$ such that

$$L_\rho \geq K L_j \ .$$

The proof of theorem B will also use the Blaschke metric h^B on Ω_ρ as an intermediate object. Indeed, one starts by deducing from the Main Lemma that the length spectrum of ρ with respect to h^B is uniformly smaller than the length spectrum of ρ with respect to the Hilbert metric on Ω_ρ (see Corollary 3.2). Then one considers the unique complete metric h^P on Ω_ρ which is conformal to h^B and whose curvature is -1. Calabi's theorem (Theorem 0.5) together with the classical Ahlfors–Schwarz–Pick lemma (Lemma 3.3) imply the inequality

$$h^B \geq h^P \ .$$

The action of ρ on (Ω_ρ, h^P) is thus isometrically conjugated to a Fuchsian representation j acting on $\mathbb{H}^2$ whose length spectrum will satisfy

$$L_j \leq L_\rho \ .$$

0.4. **Hitchin representations in higher dimensions.** We finish this introduction by mentioning a possible generalization of Theorem B to Hitchin representations in higher dimensions.

Let us still denote by S a closed connected oriented surface of genus greater than 1 and by Γ its fundamental group. A representation of Γ into $\mathrm{PSL}(2,\mathbb{R})$ induces in a natural way a representation into $\mathrm{PSL}(n,\mathbb{R})$ by post-composing with the irreducible representation

$$\iota_n : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(n,\mathbb{R}) \ .$$

By extension, we will say that a representation of Γ into $\mathrm{PSL}(n,\mathbb{R})$ is Fuchsian if it decomposes as $\iota_n \circ j$, with j a Fuchsian representation of Γ into $\mathrm{PSL}(2,\mathbb{R})$.

In [16], Hitchin described the connected components of Fuchsian representations in the space of all representations of Γ into $\mathrm{PSL}(n,\mathbb{R})$. For this reason, representations of Γ into $\mathrm{PSL}(n,\mathbb{R})$ that can be continuously deformed into Fuchsian representations are called *Hitchin representations.*

In the special case where $n = 3$, Choi and Goldman gave a geometric interpretation of Hitchin representations:

Theorem 0.8 (Choi–Goldman, [9]). *A representation of the fundamental group of a closed connected oriented surface of genus greater than* 1 *into* $\mathrm{PSL}(3, \mathbb{R})$ *is Hitchin if and only if it divides a proper open convex domain in* $\mathbb{R}\mathbf{P}^2$.

This explains the terminology we used in the previous paragraph.

One can define the length spectrum of a representation of Γ into $\mathrm{PSL}(n, \mathbb{R})$ by looking at the action of $\mathrm{PSL}(n, \mathbb{R})$ on its symmetric space $\mathrm{PSL}(n, \mathbb{R})/\mathrm{PSO}(n)$. Indeed, this symmetric space carries several $\mathrm{PSL}(n, \mathbb{R})$-invariant Finsler metrics (all of which are bi-Lipschitz equivalent to the symmetric Riemannian metric). Among them, there is a unique $\mathrm{PSL}(n, \mathbb{R})$-invariant Finsler metric such that for any $\lambda_1 > \ldots > \lambda_n$ with $\sum_i \lambda_i = 0$ the diagonal matrix

$$\begin{pmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{pmatrix}$$

has translation length $\frac{1}{2}(\lambda_1 - \lambda_n)$. We shall then denote by L_ρ the length spectrum of a representation $\rho : \Gamma \to \mathrm{PSL}(n, \mathbb{R})$ with respect to this particular Finsler metric. In the case of Hitchin representations into $\mathrm{PSL}(3, \mathbb{R})$, one recovers the length spectrum of ρ considered as a representation of Γ into the isometry group of (Ω_ρ, h^H).

In a recent work, Lee and Zhang proved that Hitchin representations satisfy the following property:

Theorem 0.9 (Lee–Zhang, [19]). *If* γ, γ' *are two curves on* S *that are not homotopic to disjoint curves, then, for any Hitchin representation* $\rho : \pi_1(S) \to \mathrm{PSL}(n, \mathbb{R})$, *one has*

$$(\exp(2L_\rho(\gamma)) - 1) \cdot \big(\exp(2L_\rho(\gamma')) - 1\big) > 1 \ .$$

This result is a slightly weaker version of the classical *collar lemma* for Fuchsian representations. Moreover, Lee and Zhang conjectured the following property of Hitchin representations:

Conjecture (Lee–Zhang). *For any Hitchin representation* $\rho : \Gamma \to \mathrm{PSL}(n, \mathbb{R})$, *there exists a Fuchsian representation* $j : \Gamma \to \mathrm{PSL}(n, \mathbb{R})$ *such that*

$$L_j \leq L_\rho \ .$$

Theorem B answers positively to this conjecture when $n = 3$. As Lee and Zhang pointed out, this implies a sharper version of the collar lemma:

Corollary C (Collar lemma for Hitchin representations into $\mathrm{PSL}(3, \mathbb{R})$). *Let* γ, γ' *be two curves on* S *that are not homotopic to disjoint curves. Then, for any Hitchin representation* ρ *into* $\mathrm{PSL}(3, \mathbb{R})$, *one has*

$$\sinh(L_\rho(\gamma)/2) \cdot \sinh(L_\rho(\gamma')/2) > 1 \ .$$

Labourie explained to us that the conjecture of Lee and Zhang cannot hold anymore for $n \geq 4$ as a consequence of several recent works on Hitchin representations (see [19, Subsection 3.3]). The impossibility comes from Hitchin representations into $\mathrm{PSp}(2k, \mathbb{R})$ and $\mathrm{PSO}(k, k+1)$. This leads us to modify the conjecture of Lee and Zhang:

Conjecture.

- *For any Hitchin representation $\rho : \Gamma \to \mathrm{PSL}(2k, \mathbb{R})$, there exists a Hitchin representation $j : \Gamma \to \mathrm{PSp}(2k, \mathbb{R})$ which satisfies*
$$L_j \leq L_\rho \text{ , and}$$
- *for any Hitchin representation $\rho : \Gamma \to \mathrm{PSL}(2k+1, \mathbb{R})$, there exists a Hitchin representation $j : \Gamma \to \mathrm{SO}_0(k, k+1)$ which satisfies*
$$L_j \leq L_\rho \ .$$

Note that the irreducible representation of $\mathrm{PSL}(2, \mathbb{R})$ into $\mathrm{PSL}(3, \mathbb{R})$ identifies $\mathrm{PSL}(2, \mathbb{R})$ with $\mathrm{SO}_0(2, 1)$. This modified conjecture would thus be a generalization of Theorem B.

0.5. **Content of the article.** In Section 1, we recall the definitions of Blaschke and Hilbert metrics. We then prove the Main Lemma and Theorem A in Section 2. Finally, in Section 3, we focus on representations of surface groups into $\mathrm{PSL}(3, \mathbb{R})$. We prove Theorem B and make several remarks concerning the behaviour of the length spectrum of Hitchin representations that are "far from being Fuchsian". Courtois brought to our attention that these remarks are essentially contained in the recent paper [23] by Nie.

0.6. **Acknowledgements.** I am thankful to Yves Benoist for enlightening discussions, to Gilles Courtois for pointing out the paper of Xin Nie, and to Constantin Vernicos for his careful reading of a previous version of this paper. Finally, I would like to thank the referees for several thoughtful comments that helped improve the quality of this paper.

1. Hilbert and Blaschke metrics

1.1. **The Hilbert metric.** Fix a proper open convex domain Ω in $\mathbb{R}\mathbf{P}^n$. Recall that the *cross-ratio* of four collinear points x_1, x_2, x_3, x_4 in $\mathbb{R}\mathbf{P}^n$ is the number $t = [x_1, x_2, x_3, x_4]$ such that (x_1, x_2, x_3, x_4) is mapped to $(0, 1, \infty, t)$ by a homography.

Given any two distinct points x and y in Ω, let a and b be the intersection points of the projective line passing through x and y with the boundary of Ω (so that x is between a and y).

Definition 1.1. The *Hilbert distance* between x and y is defined by
$$d^H_\Omega(x, y) = \frac{1}{2} \log[a, x, b, y] \ .$$

It is well-known (although not trivial) that this formula indeed defines a distance on Ω. Moreover, this distance is infinitesimally generated by a Finsler metric. More precisely, let us define
$$h^H_{\Omega,x}(u) = \frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}t^2}_{|t=0} d^H_\Omega(x, x+tu)^2$$

for any point x in Ω and any vector $u \in T_x(\Omega)$. Then, at any point x, the square root of the function $h^H_{\Omega,x}$ is a norm on $T_x\Omega$, and for any $y, z \in \Omega$, we have

$$d^H_\Omega(y,z) = \inf_\gamma \int_0^1 \sqrt{h^H_\Omega(\gamma'(t))}\mathrm{d}t \ ,$$

where the infimum is taken over all $\mathcal{C}^1$ paths from y to z. The function h^H_Ω is called the *Hilbert metric* of Ω.

The Hilbert metric is a projective invariant: if Ω is a proper open convex domain in $\mathbb{R}\mathbf{P}^n$ and g is a projective transformation, then the Hilbert metric of $g(\Omega)$ is $g_* h^H_\Omega$. In particular, every projective transformation g satisfying $g(\Omega) = \Omega$ is an isometry for (Ω, h^H_Ω). When Ω is an ellipsoid, it is preserved by a subgroup of $\mathrm{PSL}(n+1,\mathbb{R})$ conjugated to $\mathrm{PSO}(n,1)$, which implies that h^H_Ω is Riemannian and (Ω, h^H_Ω) is isometric to the hyperbolic space $\mathbb{H}^n$.

From now on we will write h^H instead of h^H_Ω without risk of confusion. The shape of the unit ball for the Hilbert metric at a point $x \in \Omega$ mirrors in some sense the shape of the boundary of Ω viewed from x. This implies that h^H often has low regularity and is never Riemannian unless Ω is an ellipsoid. In many intersesting examples it is therefore almost impossible to use only analysis to study the metric h^H. This motivates the introduction of an auxiliary Riemannian metric, called the *Blaschke metric*, which is also projectively invariant and has better differential properties.

1.2. **The Blaschke metric.** The price to pay for defining the Blaschke metric is a construction which is much more elaborate and relies on an existence theorem for solutions of certain Monge–Ampère equations. Let Ω be a proper open convex domain in $\mathbb{R}\mathbf{P}^n$. Then the Blaschke metric is the second fundamental form of a certain smooth hypersurface in $\mathbb{R}^{n+1}$ asymptotic to the cone over Ω, called the *affine sphere*. The definition we give here is not the usual one and is adapted from [3, Definition 2.1].

Denote by L the restriction to Ω of the tautological $\mathbb{R}$-line bundle over $\mathbb{R}\mathbf{P}^n$ and by ξ a nowhere vanishing smooth section of L. One can see $\xi(\Omega)$ as a smooth hypersurface in $\mathbb{R}^{n+1}$ transverse to the lines passing through the origin. Let E denote the pull-back of the tangent bundle of $\mathbb{R}^{n+1}$ by ξ. Then E writes as a direct sum

$$E = T\Omega \oplus L \ ,$$

where $T\Omega$ is identified with its image by $\mathrm{d}\xi$ (see Figure 1).

Now, the bundle E inherits from $\mathbb{R}^{n+1}$ a volume form ω and a flat linear connection ∇. For any vector fields X and Y on Ω, one can write

$$\nabla_X Y = \nabla^\xi_X Y + h(X,Y)\xi \ ,$$

where ∇^ξ is a linear connection on $T\Omega$ and h is a symmetric bilinear form on Ω.

Remark 1.2. In practical terms, the second fundamental form h can be computed in the following way: if $u : (-\varepsilon, \varepsilon) \to \Omega$ is a smooth curve, then one can write

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}_{|t=0}\xi(u(t)) = A\xi(u(0)) + Bv$$

for some constants A and B, and where v is a tangent vector to $\xi(\Omega)$ at $\xi(u(0))$. The number A only depends on the first derivative of u at 0 and we have

$$h\left(\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}u(t)\right) = A \ .$$

We say that the hypersurface $\xi(\Omega)$ has *positive Hessian* if h is positive definite and is *proper* if the map $\xi : \Omega \to \mathbb{R}^{n+1}$ is proper.

Definition 1.3. The hypersurface $\xi(\Omega)$ is a hyperbolic *affine sphere* of affine curvature -1 asymptotic to Ω if $\xi(\Omega)$ is proper, has positive Hessian, and

$$|\omega(X_1, \ldots, X_n, \xi(x))| = 1$$

for any point $x \in \Omega$ and any orthonormal basis $(X_1, \ldots, X_n)$ of $(T_x(\Omega), h_x)$.

Finding a hyperbolic affine sphere asymptotic to Ω boils down to solving some Monge-Ampère equation on Ω with boundary conditions. This allowed Cheng and Yau to prove the following important result:

Theorem 1.4 (Cheng–Yau, [8])**.** *For any proper open convex domain Ω in $\mathbb{R}\mathbf{P}^n$, there is a unique (up to reflection through the origin) hyperbolic affine sphere $\xi(\Omega) \subset (\mathbb{R}^{n+1}, \omega)$ asymptotic to Ω.*

The metric h on Ω associated with the unique affine sphere is then called the *Blaschke metric* and is denoted by h^B. Moreover, Cheng and Yau proved that the affine sphere and the Blaschke metric are analytic. The metric h^B defines a distance d^B on Ω that we call the *Blaschke distance.*

Remark 1.5. The hyperbolic affine sphere asymptotic to Ω depends on the choice of the volume ω on $\mathbb{R}^{n+1}$ in a very simple way. If ω is multiplied by $\lambda > 0$, then the affine sphere is transformed by the homothety centered at the origin of ratio $\lambda^{-\frac{1}{n+1}}$ and the Blaschke metric is unchanged.

Proposition 1.6. *When the convex Ω is an ellipsoid, the hyperbolic affine sphere asymptotic to Ω is a hyperboloid and the Blaschke metric h^B coincides with the Hilbert metric h^H (see Figure 1).*

Proof. When Ω is an ellipsoid, the Blaschke metric and the hyperbolic affine sphere asymptotic to Ω are both preserved by the group of projective transformations of Ω, which is isomorphic to $\mathrm{SO}(n, 1)$. The conclusion follows from the fact that the orbits of $\mathrm{SO}(n, 1)$ on $\mathbb{R}^{n+1}$ are hyperboloids, and that the action on Ω is transitive on unit tangent vectors for the Blaschke metric. □

2. Comparison between Hilbert and Blaschke metrics

In this section, we prove the Main Lemma and obtain theorem A as a consequence.

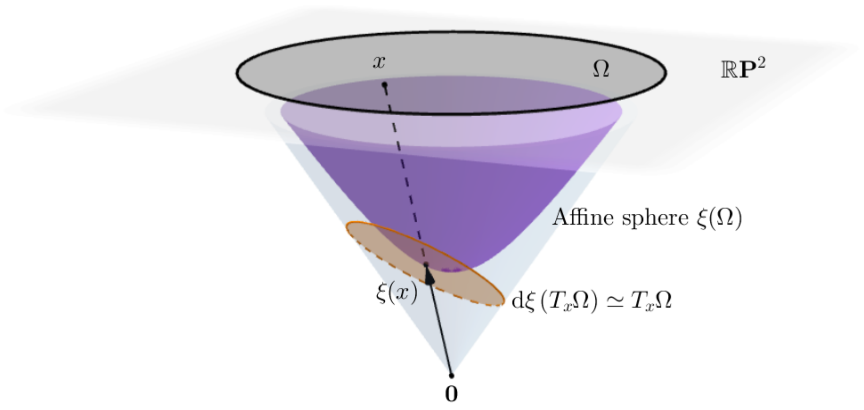

FIGURE 1. When Ω is a disc in $\mathbb{R}\mathbf{P}^2$, the hyperbolic affine sphere asymptotic to Ω is a hyperboloid.

2.1. **Proof of the Main Lemma.** Let Ω be a proper open convex domain in $\mathbb{R}\mathbf{P}^n$. If u is a non-zero vector in $\mathbb{R}^{n+1}$, we denote by $[u]$ its projection in $\mathbb{R}\mathbf{P}^n$.

Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be two non-zero vectors in $\mathbb{R}^{n+1}$ whose projections $[\mathbf{e}_1]$ and $[\mathbf{e}_2]$ in $\mathbb{R}\mathbf{P}^n$ are distinct points of $\partial\Omega$. We now restrict to the plane generated by these directions.

We parametrize the projective segment between $[\mathbf{e}_1]$ and $[\mathbf{e}_2]$ by

$$\left\{[e^t\mathbf{e}_1 + e^{-t}\mathbf{e}_2], t \in \mathbb{R}\right\} \ .$$

The intersection of the hyperbolic affine sphere asymptotic to Ω with the plane generated by e_1 and e_2 is thus parametrized by

$$\left\{u(t) = e^{t+\alpha(t)}\mathbf{e}_1 + e^{-t+\alpha(t)}\mathbf{e}_2, t \in \mathbb{R}\right\}$$

for some smooth function $\alpha : \mathbb{R} \to \mathbb{R}$.

One easily verifies that for any $t_1, t_2 \in \mathbb{R}$ we have

$$d^H([u(t_1)], [u(t_2)]) = |t_1 - t_2|$$

which yields

$$h^H\left(\frac{\mathrm{d}}{\mathrm{d}t}[u(t)]\right) = 1$$

for any $t \in \mathbb{R}$. In other words, $t \mapsto [u(t)]$ is a geodesic for the Hilbert metric on Ω.

Let us now compute $h^B\left(\frac{\mathrm{d}}{\mathrm{d}t}[u(t)]\right)$.

Lemma 2.1. *For any $t \in \mathbb{R}$, we have*

$$h^B\left(\frac{\mathrm{d}}{\mathrm{d}t}[u(t)]\right) = \alpha''(t) - \alpha'^2(t) + 1 \ .$$

Proof. By definition of h^B (see Remark 1.2), we can write

$$u''(t) = h^B\left(\frac{\mathrm{d}}{\mathrm{d}t}[u(t)]\right)u(t) + B(t)u'(t)\ .$$

On the other hand, we compute

$$u'(t) = (\alpha'(t)+1)e^{t+\alpha(t)}\mathbf{e}_1 + (\alpha'(t)-1)e^{-t+\alpha(t)}\mathbf{e}_2$$

and

$$u''(t) = \left(\alpha''(t) + (\alpha'(t)+1)^2\right)e^{t+\alpha(t)}\mathbf{e}_1 + \left(\alpha''(t) + (\alpha'(t)-1)^2\right)e^{-t+\alpha(t)}\mathbf{e}_2\ .$$

To obtain the coordinates of $u''(t)$ in the basis $(u(t), u'(t))$, we invert the matrix

$$A(t) = \begin{pmatrix} e^{t+\alpha(t)} & (\alpha'(t)+1)e^{t+\alpha(t)} \\ e^{-t+\alpha(t)} & (\alpha'(t)-1)e^{-t+\alpha(t)} \end{pmatrix}\ .$$

We get

$$A^{-1}(t) = \frac{1}{-2e^{2\alpha(t)}}\begin{pmatrix} (\alpha'(t)-1)e^{-t+\alpha(t)} & -(\alpha'(t)+1)e^{t+\alpha(t)} \\ -e^{-t+\alpha(t)} & e^{t+\alpha(t)} \end{pmatrix}\ .$$

The coordinates of $u''(t)$ in the basis $(u(t), u'(t))$ are thus given by

$$A^{-1}(t)\begin{pmatrix} \left(\alpha''(t) + (\alpha'(t)+1)^2\right)e^{t+\alpha(t)} \\ \left(\alpha''(t) + (\alpha'(t)-1)^2\right)e^{-t+\alpha(t)} \end{pmatrix}\ ,$$

which yields

$$\begin{aligned} h^B\left(\frac{\mathrm{d}}{\mathrm{d}t}[u(t)]\right) &= -\frac{1}{2}\left[(\alpha'(t)-1)\left(\alpha''(t) + (\alpha'(t)+1)^2\right)\right. \\ &\quad \left. - (\alpha'(t)+1)\left(\alpha''(t) + (\alpha'(t)-1)^2\right)\right] \\ &= \alpha''(t) - \alpha'^2(t) + 1\ . \end{aligned}$$

□

Let us now prove that α' is bounded by 1. More precisely, we show that

Proposition 2.2. *If we have $h^B \geq Ch^H$ for some constant $C \in [0, 1]$, then*

$$\alpha'^2 \leq 1 - C\ .$$

In order to prove this proposition, we will use the following classical lemma:

Lemma 2.3. *Let f and g be two functions of class $\mathcal{C}^1$ on an interval $[t_0, t_1)$, and let $H : \mathbb{R} \to \mathbb{R}$ be a smooth function. Suppose that we have*

$$f'(t) = H(f(t))$$

and

$$g'(t) \geq H(g(t))$$

for all $t \in [t_0, t_1)$.

If $g(t_0) \geq f(t_0)$, then $g(t) \geq f(t)$ for all $t \in [t_0, t_1)$.

Proof. For any $\varepsilon > 0$ small enough, denote by f_ε the unique solution on $[t_0, t_1)$ of the equation

$$f'_\varepsilon = H(f_\varepsilon) - \varepsilon$$

with initial condition $f_\varepsilon(t_0) = f(t_0) - \varepsilon$. Assume by contradiction that $g(t) \leq f_\varepsilon(t)$ for some $t \in [t_0, t_1)$. Since $g(t_0) > f(t_0) - \varepsilon = f_\varepsilon(t_0)$, one can define

$$t_m = \min\{t \in [t_0, t_1) \mid g(t) = f_\varepsilon(t)\} > t_0 \ .$$

For all $t \in [t_0, t_m]$, we have

$$g(t) \geq f_\varepsilon(t) \ ,$$

which implies that

$$g'(t_m) \leq f'_\varepsilon(t_m) \ .$$

This contradicts the fact that

$$\begin{aligned} g'(t_m) &\geq H(g(t_m)) \\ &> H(g(t_m)) - \varepsilon = H(f_\varepsilon(t_m)) - \varepsilon = f'_\varepsilon(t_m) \ . \end{aligned}$$

Therefore, $g > f_\varepsilon$ on $[t_0, t_1)$. Taking the limit when ε goes to 0, we obtain

$$g \geq f$$

on $[t_0, t_1)$. □

Proof of Proposition 2.2. Since $h^B\left(\frac{\mathrm{d}}{\mathrm{d}t}[u(t)]\right) = \alpha'' - \alpha'^2 + 1$ by Lemma 2.1 and since $h^H\left(\frac{\mathrm{d}}{\mathrm{d}t}[u(t)]\right) = 1$, we have

$$\alpha'' - \alpha'^2 + 1 \geq C \ ,$$

which we can write

$$\alpha'' \geq \alpha'^2 - \beta^2 \ , \tag{1}$$

where $\beta = \sqrt{1 - C}$.

Assume by contradiction that

$$\alpha'(t_0) > \beta$$

for some $t_0 \in \mathbb{R}$. By Lemma 2.3, Inequality (1) implies that the function f defined as the unique solution of the ordinary differential equation

$$f' = f^2 - \beta^2$$

with initial condition

$$f(t_0) = \alpha'(t_0)$$

satisfies

$$f(t) \leq \alpha'(t)$$

for all $t \geq t_0$.

It is now a simple exercise to compute explicitly the function f. When $\beta > 0$, one finds

$$f(t) = \beta \left(\frac{1 + De^{2\beta(t - t_0)}}{1 - De^{2\beta(t - t_0)}} \right) \ ,$$

where

$$D = \frac{\alpha'(t_0) - \beta}{\alpha'(t_0) + \beta} \in (0, 1) \ .$$

In particular, $f(t)$ goes to $+\infty$ when t goes to

$$t_{max} = t_0 - \frac{1}{2\beta}\log(D) > t_0 \ .$$

This contradicts the fact that

$$f(t) \leq \alpha'(t) \leq \sup_{t_0 \leq u \leq t_{max}} \alpha'(u)$$

for all $t \in [t_0, t_{max})$.

Similarly, if we had $\alpha'(t_0) < -\beta$ for some $t_0 \in \mathbb{R}$, we would obtain that $f(t)$ goes to $-\infty$ as t goes to some time $t_{min} < t_0$, which would contradict the fact that α' is bounded on $[t_{min}, t_0]$. We thus conclude that

$$\alpha'^2 \leq \beta^2 = 1 - C \ .$$

When $\beta = 0$, we find

$$f(t) = \frac{f(t_0)}{1 - f(t_0)(t - t_0)}$$

and the rest of the proof is the same. □

Finally, we prove a sharper version of the Main Lemma.

Lemma 2.4. *Let $C \in [0,1]$. If we have $h^B \geq C h^H$, then*

$$d^B(x,y) \leq d^H(x,y) + \sqrt{1-C}$$

for all points $x, y \in \Omega$.

Since the Blaschke metric is positive, we immediately obtain $d^B \leq d^H + 1$ by applying Lemma 2.4 with $C = 0$. Nevertheless, thanks to Benoist–Hulin's theorem (Theorem 0.1), Lemma 2.4 shows that the latter inequality is actually strict, which proves the Main Lemma.

Proof of Lemma 2.4. Given $t_1 < t_2 \in \mathbb{R}$, the distance $d^B([u(t_1)], [u(t_2)])$ is bounded from above by the length of the path $\{[u(t)], t \in [t_1, t_2]\}$ with respect to the Blaschke metric. We thus have

$$\begin{aligned}
d^B([u(t_1)], [u(t_2)])^2 &\leq \left(\int_{t_1}^{t_2} \sqrt{h^B\left(\frac{\mathrm{d}}{\mathrm{d}t}[u(t)]\right)}\mathrm{d}t \right)^2 \\
&\leq (t_2 - t_1) \int_{t_1}^{t_2} h^B\left(\frac{\mathrm{d}}{\mathrm{d}t}[u(t)]\mathrm{d}t\right) \quad \text{(by Cauchy–Schwarz inequality)} \\
&\leq (t_2 - t_1) \int_{t_1}^{t_2} (\alpha''(t) - \alpha'^2(t) + 1)\mathrm{d}t \quad \text{(by Lemma 2.1)} \\
&\leq (t_2 - t_1)\left(t_2 - t_1 + \alpha'(t_2) - \alpha'(t_1)\right) \ .
\end{aligned}$$

Now, since $\alpha'(t_2) - \alpha'(t_1) \leq 2\beta$ by Proposition 2.2, we obtain

$$\begin{aligned}
d^B([u(t_1)], [u(t_2)]) &\leq (t_2 - t_1)\sqrt{1 + 2\frac{\beta}{t_2 - t_1}} \\
&\leq t_2 - t_1 + \beta = d^H([u(t_1)], [u(t_2)]) + \beta \ .
\end{aligned}$$

(We used the classical inequality: $\sqrt{1+x} \leq 1 + \frac{x}{2}$.) We have thus proved that

$$d^B(x,y) \leq d^H(x,y) + \sqrt{1-C}$$

for any two points x and y on the projective segment joining $[\mathbf{e}_1]$ and $[\mathbf{e}_2]$.

Finally, since $[\mathbf{e}_1]$ and $[\mathbf{e}_2]$ were arbitrarily chosen in $\partial\Omega$, this concludes the proof of Lemma 2.4 and hence that of the Main Lemma. □

Remark 2.5. So far, we didn't really use that h^B is the Blaschke metric on Ω. Indeed, Lemma 2.4 still holds if we replace h^B by any metric defined as the second fundamental form of some hypersurface with positive Hessian asymptotic to Ω. However, we don't know any other metric than h^B to which it would be interesting to apply this lemma.

Lemma 2.4 is "sharper" than the Main Lemma in the sense that it explicitly links the upper bound of d^B with the lower bound of h^B. If we apply it to the particular case $C = 1$, we obtain the following corollary:

Corollary 2.6. *Let Ω be a proper open convex domain in $\mathbb{R}\mathbf{P}^n$. Denote by h^H and h^B its Hilbert and Blaschke metrics, respectively. If we have*

$$h^B \geq h^H \ ,$$

then

$$h^B = h^H$$

and Ω is an ellipsoid.

Proof. By lemma 2.4, if we have $h^B \geq h^H$, then $d^B \leq d^H$. But this yields $h^B \leq h^H$ by the very definition of d^H and d^B. Hence we get $h^B = h^H$. The Hilbert metric h^H is therefore Riemannian, which implies that Ω is an ellipsoid (see Subsection 1.1).

□

2.2. **Volume entropy of Hilbert and Blaschke metrics.** Recall that the volume entropy of the Hilbert metric h^H is defined by

$$\mathcal{H}(h^H) = \limsup_{R\to+\infty} \frac{1}{R} \log \mathbf{Vol}\left(B^H(o,R)\right) \ ,$$

where o is any base point in Ω and **Vol** is the volume with respect to any uniform volume form *vol* on Ω. According to Theorem 0.1, one can choose *vol* to be the volume form induced by the Blaschke metric h^B.

Lemma 2.7. *We have*

$$\mathcal{H}(h^H) \leq \mathcal{H}(h^B) \ .$$

Proof. For any $R > 0$, denote by $B^H(o,R)$ and $B^B(o,R)$ the balls of radius R about o with respect to the Hilbert and Blaschke metrics, respectively. According to the Main Lemma, we have

$$B^H(o,R) \subset B^B(o,R+1) \ .$$

Therefore

$$\mathbf{Vol}(B^H(o,R)) \leq \mathbf{Vol}(B^B(o,R+1)) \ ,$$

and hence

$$\begin{aligned}\mathcal{H}(h^H) &= \limsup_{R\to+\infty} \frac{1}{R}\log \mathbf{Vol}\left(B^H(o,R)\right)\\ &\leq \limsup_{R\to+\infty} \frac{1}{R+1}\log \mathbf{Vol}\left(B^B(o,R+1)\right)\\ &\leq \mathcal{H}(h^B)\ .\end{aligned}$$

□

Now, in order to prove Theorem A, let us recall a famous theorem of Bishop:

Theorem 2.8 (Bishop)**.** *If g is a smooth Riemannian metric on a manifold M whose Ricci curvature is bounded from below by $-(n-1)$, then for all R,*

$$\mathbf{Vol}_g(B_g) \leq \mathbf{Vol}_{\mathbb{H}^n}(B_{\mathbb{H}^n})\ ,$$

where B_g and $B_{\mathbb{H}^n}$ denote arbitrary balls of radius R in M and $\mathbb{H}^n$, respectively.

Since Calabi's theorem (Theorem 0.5) insures that the Ricci curvature of the metric h^B is bounded from below by $-(n-1)$, Bishop's theorem (Theorem 2.8) implies that $\mathcal{H}(h^B) \leq n-1$ (the volume entropy of $\mathbb{H}^n$). Since $\mathcal{H}(h^H) \leq \mathcal{H}(h^B)$ by Lemma 2.7, this finishes the proof of Theorem A.

3. The two-dimensional case and surface group representations

In this section, we make use of the Main Lemma to study the length spectrum of Hitchin representations of surface groups into $\mathrm{PSL}(3,\mathbb{R})$. We prove Theorem B and describe the behaviour of this length spectrum "far away" from the Fuchsian locus.

Let us first recall the following definition:

Definition 3.1. The *translation length* of an isometry g of a metric space (X,d) is the number

$$l(g) = \lim_{n\to+\infty} \frac{1}{n} d(x, g^n\cdot x)\ ,$$

where x is any point of X.

If Ω is a proper open convex domain in $\mathbb{R}\mathbf{P}^n$ and g a projective transformation which satisfies $g(\Omega)=\Omega$, let us denote by $l^H(g)$ and $l^B(g)$ the translation lengths of g considered as an isometry of (Ω, h^H) and (Ω, h^B), respectively. As a consequence of the Main Lemma, we easily obtain the following corollary:

Corollary 3.2. *Let Ω be a proper open convex domain in $\mathbb{R}\mathbf{P}^n$ and g a projective transformation which satisfies $g(\Omega)=\Omega$. Then we have*

$$l^B(g) \leq l^H(g)\ .$$

Proof. By the Main Lemma, we have

$$\frac{1}{n}d^B(x, g^n \cdot x) \leq \frac{1}{n}d^H(x, g^n \cdot x) + \frac{1}{n}$$

for any positive integer n. Then, passing to the limit when n goes to $+\infty$, we get

$$l^B(g) \leq l^H(g) \ .$$

□

Let us now specialize this result to divisible proper open convex domains in dimension 2.

3.1. **Proof of Theorem B.** Fix a closed connected oriented surface S of genus greater than 1. Denote by Γ its fundamental group and consider a Hitchin representation

$$\rho : \Gamma \to \mathrm{PSL}(3, \mathbb{R}) \ .$$

According to Choi-Goldman's theorem (Theorem 0.8), $\rho(\Gamma)$ acts freely, properly discontinuously and cocompactly on a proper open convex domain $\Omega_\rho \subset \mathbb{R}\mathbf{P}^2$.

The Blaschke metric h^B on Ω_ρ is preserved by ρ and thus induces a Riemannian metric on

$$\Omega_\rho/\rho(\Gamma) \cong S$$

that we still denote by h^B. By Poincaré–Koebe's uniformization theorem, there exists a unique complete Riemannian metric h^P on Ω_ρ conformal to h^B and of constant curvature -1. Moreover, this metric is also invariant under the action of $\rho(\Gamma)$. Let us also denote by h^P the induced metric on $\Omega_\rho/\rho(\Gamma)$.

We now recall the following classical fact, sometimes refered to as the Ahlfors–Schwarz–Pick lemma (see [25] for a fairly general version).

Lemma 3.3. *Let h and h' be two conformal Riemannian metrics on a closed surface. If we have $\kappa(h) \leq \kappa(h') \leq 0$, then either $h' = h$ or there exists a constant $K > 1$ such that $h' \geq Kh$.*

This lemma applies in particular to $h = h^P$ and $h' = h^B$ since the curvature of h^B is bounded between -1 and 0 according to Theorem 0.5.

On the other hand, since the convex Ω_ρ with the metric h^P is locally isometric to the hyperbolic plane $\mathbb{H}^2$ and since ρ is injective and acts properly discontinuously on Ω_ρ, we can find a Fuchsian representation $j : \Gamma \to \mathrm{PSL}(2, \mathbb{R})$ for which there is a (ρ, j)-equivariant isometry from (Ω_ρ, h^P) to $\mathbb{H}^2$.

Lemma 3.3 together with Corollary 3.2 then imply the following:

Corollary 3.4. *Either ρ is Fuchsian or there exists a constant $K > 1$ such that*

$$L_\rho \geq K L_j \ ,$$

where L_ρ denotes the length spectrum of ρ with respect to the Hilbert metric on Ω_ρ.

Proof. If we have $h^B = h^P$, then Ω_ρ is a disc (this can be deduced for instance from [4]) and ρ is itself Fuchsian. Otherwise, there is a constant $K > 1$ such that $h^B \geq Kh^P$ by Lemma 3.3. Let γ be any element of Γ and denote by $l^P(\gamma)$, $l^B(\gamma)$ and $l^H(\gamma)$ the translation lengths of $\rho(\gamma)$ with respect to the metrics h^P, h^B and h^H, respectively. We then have

$$l^B(\gamma) \geq Kl^P(\gamma) = KL_j(\gamma)$$

since the action of $\rho(\Gamma)$ on (Ω_ρ, h^P) is isometrically conjugated to the action of $j(\Gamma)$ on $\mathbb{H}^2$. On the other hand, Corollary 3.2 yields

$$l^B(\gamma) \leq l^H(\gamma) = L_\rho(\gamma)\ .$$

Thus, we obtain

$$L_\rho \geq KL_j\ .$$

□

This concludes the proof of Theorem B. Finally, let us prove Corollary C.

Proof of Corollary C. Let γ and η be two elements of $\Gamma = \pi_1(S)$ represented by two curves that are not freely homotopic to disjoint curves. Let $\rho : \Gamma \to \mathrm{PSL}(3, \mathbb{R})$ be a Hitchin representation. By Theorem B, there exists a Fuchsian representation j such that $L_j \leq L_\rho$. The classical collar lemma (see [17]) then asserts that

$$\sinh\left(\frac{L_j(\gamma)}{2}\right) \cdot \sinh\left(\frac{L_j(\eta)}{2}\right) > 1\ ,$$

which yields

$$\sinh\left(\frac{L_\rho(\gamma)}{2}\right) \cdot \sinh\left(\frac{L_\rho(\eta)}{2}\right) > 1\ .$$

□

3.2. **Asymptotic behaviour of the length spectrum.** We end this section with a description of the asymptotic behaviour of the length spectrum away from the Fuchsian locus. The following results are consequences of the work of Loftin and Benoist–Hulin (see [22] and [3]). As Courtois pointed out to us, they are essentially contained in the recent paper [23] by Nie (although Nie actually focuses on the entropy of Hitchin representations into $\mathrm{PSL}(3, \mathbb{R})$).

Let Ω be a proper open convex domain in $\mathbb{R}\mathbf{P}^n$ and $\xi(\Omega)$ the hyperbolic affine sphere asymptotic to Ω. With the notations of Subsection 1.2, the linear connection ∇^ξ does not preserve h^B in general. The difference between ∇^ξ and the Levi–Civita connection of h^B is called the *Pick tensor* of Ω. Denote it by P, and define

$$A(X, Y, Z) = h^B(P(X, Y), Z)\ .$$

Then the tensor A is symmetric in X, Y, Z.

Proposition 3.5 (see [3], Lemma 4.8)**.** *Let Ω be a proper open convex domain in $\mathbb{R}\mathbf{P}^2$ equipped with the conformal structure induced by h^B. Then the tensor A is the real part of a holomorphic cubic differential, called the* Pick form *of Ω.*

Now, let S be a closed connected surface of genus greater than 1, Γ its fundamental group, and ρ a Hitchin representation of Γ into $\mathrm{PSL}(3,\mathbb{R})$. Let $\Omega_\rho \subset \mathbb{R}\mathbf{P}^2$ be the proper open convex domain divided by ρ. Then the Pick form of Ω_ρ and the conformal class of the Blaschke metric on Ω_ρ are both preserved by $\rho(\Gamma)$ and thus induce a conformal structure and a holomorphic cubic differential on $\Omega_\rho/\rho(\Gamma) \cong S$. Loftin and Labourie proved that this actually gives a parametrization of the space of Hitchin representations of Γ into $\mathrm{PSL}(3,\mathbb{R})$:

Theorem 3.6 (Labourie [18], Loftin [20])**.** *Let J be a conformal structure on S and Φ a holomorphic cubic differential on (S,J). Then there is a unique (up to conjugacy) Hitchin representation $\rho = \rho(J,\Phi)$ of Γ into $\mathrm{PSL}(3,\mathbb{R})$ for which there exists a ρ-equivariant homeomorphism from the universal cover of S to Ω_ρ sending J to the conformal class of the Blaschke metric on Ω_ρ and Φ to the Pick form of Ω_ρ.*

Remark 3.7. Generalizing this theorem of Loftin and Labourie, Benoist and Hulin recently gave in [4] a parametrization of all proper open convex domains in $\mathbb{R}\mathbf{P}^2$ that are Gromov-hyperbolic. Dumas and Wolf also gave in [14] a similar parametrization for convex polygons in $\mathbb{R}\mathbf{P}^2$.

Let J be a conformal structure on S and Φ a holomorphic cubic differential on (S,J). According to the theorem of Loftin and Labourie (Theorem 3.6), for each $t > 0$, there exists a unique (up to conjugacy) Hitchin representation $\rho_t : \Gamma \to \mathrm{PSL}(3,\mathbb{R})$ such that $(S,J,t\Phi)$ identifies with the quotient $\Omega_{\rho_t}/\rho_t(\Gamma)$ endowed with the conformal structure of its Blaschke metric and its Pick form.

Let us denote by h_t^B the Blaschke metric associated with the pair $(J,t\Phi)$. Loftin proved the following:

Theorem 3.8 (Loftin, [22])**.** *Let S' be the complement of the zeros of Φ in S. Denote by h^P the conformal Riemannian metric of curvature -1 on (S,J) and by σ_t the positive function defined by*

$$h_t^B = \sigma_t^2 h^P$$

for each $t > 0$. Then

$$\frac{\sigma_t}{t^{1/3}} \underset{t\to+\infty}{\to} 2^{1/6}|\Phi|^{1/3}$$

uniformly on every compact subset of S'. (Here, $|\Phi|$ is the pointwise norm of Φ with respect to the metric h^P.)

Using relatively simple estimates, we obtain a similar behaviour for the length spectrum of the Hitchin representation $\rho_t : \Gamma \to \mathrm{PSL}(3,\mathbb{R})$ associated with $(J,t\Phi)$.

Theorem 3.9. *For each $t > 0$, let $\rho_t : \Gamma \to \mathrm{PSL}(3,\mathbb{R})$ be the Hitchin representation associated with $(J,t\Phi)$, and let $j : \Gamma \to \mathrm{PSL}(2,\mathbb{R})$ be the Fuchsian representation uniformizing (S,J). Then there are some positive constants A and B such that*

$$At^{1/3}L_j \leq L_{\rho_t} \leq Bt^{1/3}L_j$$

for all $t > 0$.

Proof. Let us use the notations of Theorem 3.8. For each $t > 0$, the Hilbert metric h_t^H on Ω_{ρ_t} is preserved by ρ_t and thus induces a Finsler metric on $S \cong \Omega_{\rho_t}/\rho_t(\Gamma)$ that we still denote by h_t^H. By construction of ρ_t, the metric h^P on S is the unique metric of constant curvature -1 that is conformal with respect to the complex structure J. It does not depend on t and all the metrics h_t^B are in the same conformal class.

We define the *length spectrum* of the metrics h_t^H, h_t^B, and h^P as the functions L_t^H, L_t^B and L^P from Γ to $\mathbb{R}_+$ associating to an element γ the infimum of the lengths of closed curves on S freely homotopic to γ, where the lengths are measured with respect to the metrics h_t^H, h_t^B and h^P, respectively. We thus have $L_t^H = L_{\rho_t}$ and $L^P = L_j$, since the action of $\rho_t(\Gamma)$ on (Ω_{ρ_t}, h^P) is isometrically conjugated to the action of $j(\Gamma)$ on $\mathbb{H}^2$.[1]

By Benoist–Hulin's theorem (Theorem 0.1), there exists a positive constant C independent of t such that

$$\frac{1}{C} h_t^H \leq h_t^B \leq C h_t^H \ .$$

It follows that

$$\frac{1}{C} L_{\rho_t} \leq L_t^B \leq C L_{\rho_t} \ .$$

It is therefore enough to find upper and lower bounds for the length spectrum L_t^B in terms of $L^P = L_j$.

Let $\sigma_t : S \to \mathbb{R}_+^*$ be such that $h_t^B = \sigma_t^2 h^P$. The function σ_t satisfies the following partial differential equation:

$$\Delta \log(\sigma_t) = 1 + \sigma_t^2 - 2t^2 \sigma_t^{-4} |\Phi| \ ,$$

where Δ denotes the Laplace operator of the metric h^P.

Given a point x_m where σ_t achieves its maximum, we have $\Delta \log(\sigma_t)(x_m) \leq 0$. We deduce that

$$\sigma_t(x_m)^6 + \sigma_t(x_m)^4 \leq 2t^2 |\Phi(x_m)|^2 \ ,$$

which implies that

$$\sigma_t(x_m) \leq 2^{1/6} t^{1/3} \left\|\Phi\right\|_\infty^{1/3} \ ,$$

where $\left\|\Phi\right\|_\infty = \sup_{x \in S} |\Phi(x)|$.

Since σ_t achieves its maximum at x_m, we obtain

$$\sigma_t \leq 2^{1/6} t^{1/3} \left\|\Phi\right\|_\infty^{1/3}$$

on S, and therefore

$$L_t^B \leq 2^{1/6} \left\|\Phi\right\|_\infty^{1/3} t^{1/3} L^P \ .$$

To find a lower bound of L_t^B, we use the inequality

$$\sigma_t \geq t^{1/3} |\Phi|^{1/3} \ , \tag{2}$$

---

[1]Here we use the fact that for every point x in $\tilde{S} = \Omega_{\rho_t}$ and every $\gamma \in \Gamma$,

$$\inf_{y \in \tilde{S}} d(y, \gamma \cdot y) = \lim_{n \to +\infty} \frac{1}{n} d(x, \gamma^n \cdot x) \ ,$$

where d is one of the lifted distances on $\tilde{S}$ associated with h_t^H, h_t^B or h^P. Though it is not a general fact, it is well known that it holds for Hilbert metrics on open strictly convex domains in $\mathbb{R}\mathbf{P}^n$ with $\mathcal{C}^1$ boundary (see [12, Theorem 3.3]) and for Hadamard manifolds (see [1, Lemma 6.6, Point 2, page 83]).

which comes from the fact that the curvature $\kappa = -1 + 2\frac{|\Phi|^2}{\sigma^6}$ of the metric $h_t^B = \sigma_t^2 h^P$ is non-positive (see for instance [4, Proposition 3.3]).

Let us fix $\varepsilon > 0$ and chose $\eta > 0$ such that $|\Phi| \geq \varepsilon$ outside of an η-neighbourhood (for the metric h^P) of the zeros of Φ. We can choose ε and η small enough so that if x and y are two distinct zeros of the lift of Φ to the universal cover $\tilde{S}$ of S, then

$$d^P\left(B^P(x,\eta), B^P(y,\eta)\right) \geq 2\eta \ ,$$

where d^P and B^P denote the distance and balls with respect to the metric h^P.

Let γ be a closed geodesic for the metric h_t^B. Let us decompose γ into segments that stay close to a zero of Φ and segments that go from one zero to another. Indeed, we can decompose the path γ into $2n$ segments $[A_i, B_i]$ and $[B_i, A_{i+1}]$ (with $A_n = A_0$) such that, if $\tilde{A}_i, \tilde{B}_i$ denote the lifts of A_i, B_i along a lift of γ to the universal cover, we have

- $\tilde{A}_i$ and $\tilde{B}_i$ are at distance η of the same zero of $\tilde{\Phi}$,
- $\tilde{B}_i$ and $\tilde{A}_{i+1}$ are in the η-neighborood of two distinct zeros of $\tilde{\Phi}$ and the path $]B_i, A_{i+1}[$ remains outside of the η-neighborhood of the zeros of Φ.

Let d_t^B denote the distance associated with the metric h_t^B. We have

$$L_t^B(\gamma) = \sum_{i=0}^{n-1} d_t^B(\tilde{A}_i, \tilde{B}_i) + d_t^B(\tilde{B}_i, \tilde{A}_{i+1}) \geq \sum_{i=0}^{n-1} d_t^B(\tilde{B}_i, \tilde{A}_{i+1}) \ .$$

Since $|\Phi| \geq \varepsilon$ along $[B_i, A_{i+1}]$, we obtain, thanks to Inequality (2):

$$d_t^B(\tilde{B}_i, \tilde{A}_{i+1}) \geq \varepsilon^{1/3} t^{1/3} d^P(\tilde{B}_i, \tilde{A}_{i+1}) \ .$$

Finally, since $d^P(\tilde{B}_i, \tilde{A}_{i+1}) \geq 2\eta$ and $d^P(\tilde{A}_i, \tilde{B}_i) \leq 2\eta$, we have

$$d^P(\tilde{B}_i, \tilde{A}_{i+1}) \geq \frac{1}{2}\left(d^P(\tilde{A}_i, \tilde{B}_i) + d^P(\tilde{B}_i, \tilde{A}_{i+1})\right) \ .$$

Putting all this together, we obtain

$$\begin{aligned} L_t^B(\gamma) &\geq \frac{1}{2}\varepsilon^{1/3} t^{1/3} \sum_{i=0}^{n-1} d^P(\tilde{A}_i, \tilde{B}_i) + d^P(\tilde{B}_i, \tilde{A}_{i+1}) \\ &\geq \frac{1}{2}\varepsilon^{1/3} t^{1/3} L_j(\gamma) \ . \end{aligned}$$

This gives the required lower bound of the length spectrum L_t^B and concludes the proof of Theorem 3.9. □

Finally, by compactness, one can choose the same constants A and B in Theorem 3.9 for all pairs (J, Φ) for which (S, J) lies in a compact subset of the moduli space of Riemann surfaces homeomorphic to S and Φ satisfies $\|\Phi\|_J = 1$. Denoting by $\rho(J, \Phi)$ the Hitchin representation of Γ into $\mathrm{PSL}(2, \mathbb{R})$ associated with the pair (J, Φ), we get the following:

Corollary 3.10. *For any compact subset K of the moduli space of Riemann surfaces homeomorphic to S, there exists some constant $C(K) > 1$ such that* *for any pair* (J, Φ) *with* $(S, J) \in K$*, we have*

$$\frac{1}{C(K)} \|\Phi\|_J^{1/3} L_j \leq L_{\rho(J,\Phi)} \leq C(K) \|\Phi\|_J^{1/3} L_j \ .$$

Remark 3.11. In [26], Zhang constructs sequences of Hitchin representations of Γ into $\mathrm{PSL}(3, \mathbb{R})$ whose entropy goes to 0, although the translation lengths of some curves on the surface S remain bounded. According to Corollary 3.10, those sequences are associated with pairs (J_n, Φ_n), where (S, J_n) leaves every compact subset of the moduli space (otherwise the whole length spectrum of $\rho(J_n, \Phi_n)$ would go to infinity).

Remark 3.12. Loftin studied in [21] the asymptotic behaviour of Hitchin representations of Γ into $\mathrm{PSL}(3, \mathbb{R})$ associated with pairs (J_n, Φ_n), where (S, J_n) leaves every compact subset of the moduli space. It is likely that his results will give a more precise description of the behaviour of the length spectrum on the whole Hitchin component.

References

[1] Werner Ballmann, Mikhael Gromov, and Viktor Schroeder. *Manifolds of nonpositive curvature*, volume 61 of *Progress in Mathematics*. Birkhäuser, 1985.

[2] Thomas Barthelmé, Ludovic Marquis, and Andrew Zimmer. Entropy rigidity of Hilbert and Riemannian metrics. *Preprint*, 2015. arXiv:1508.07474.

[3] Yves Benoist and Dominique Hulin. Cubic differentials and finite volume convex projective surfaces. *Geometry & Topology*, 17(1):595–620, 2013.

[4] Yves Benoist and Dominique Hulin. Cubic differentials and hyperbolic convex sets. *Journal of Differential Geometry*, 98(1):1–19, 2014.

[5] Gautier Berck, Andreas Bernig, and Constantin Vernicos. Volume entropy of hilbert geometries. *Pacific Journal of Mathematics*, 245(2):201–225, 2010.

[6] Rufus Bowen. Hausdorff dimension of quasi-circles. *Publi. Math. Inst. Hautes Études Sci.*, 50(1):11–25, 1979.

[7] Eugenio Calabi. Complete affine hyperspheres I. In *Convegno di Geometria Differenziale (INDAM, Rome, 1971), Symposia Mathematica*, volume 10, pages 19–38, 1972.

[8] Shiu-Yuen Cheng and Shing-Tung Yau. On the regularity of the Monge-Ampère equation $\det\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right) = f(x, u)$. *Communications on Pure and Applied Mathematics*, 30(1):41–68, 1977.

[9] Suhyoung Choi and William M. Goldman. Convex real projective structures on closed surfaces are closed. *Proceedings of the American Mathematical Society*, 118(2):657–661, 1993.

[10] Bruno Colbois and Patrick Verovic. Hilbert geometry for strictly convex domains. *Geometriae Dedicata*, 105(1):29–42, 2004.

[11] Mickaël Crampon. Entropies of compact strictly convex projective manifolds. *J. Mod. Dyn.*, 3(4):511–547, 2009.

[12] Mickaël Crampon and Ludovic Marquis. Finitude géométrique en géométrie de Hilbert. *Ann. Inst. Fourier*, 64(6):2299–2377, 2014.

[13] Bertrand Deroin and Nicolas Tholozan. Dominating surface group representations by Fuchsian ones. *International Mathematics Research Notices*, 2015.

[14] David Dumas and Michael Wolf. Polynomial cubic differentials and convex polygons in the projective plane. *Geometric and Functional Analysis*, 25:1734–1798, 2015.

[15] David Hilbert. Über die gerade linie als kürzeste verbindung zweier punkte. *Mathematische Annalen*, 46(1):91–96, 1895.

[16] Nigel J. Hitchin. Lie groups and Teichmüller space. *Topology*, 31(3):449–473, 1992.

[17] Linda Keen. Collars on Riemann surfaces. In *Discontinuous groups and Riemann surfaces (Proc. Conf., Univ. Maryland, College Park, Md., 1973)*, pages 263–268, 1974.

[18] François Labourie. Flat projective structures on surfaces and cubic holomorphic differentials. *Pure Appl. Math. Q.*, 2007.

[19] Gye-Seon Lee and Tengren Zhang. Collar lemma for Hitchin representations. *to appear in Geometry & Topology*, 2014. arXiv:1411.2082.

[20] John C. Loftin. Affine spheres and convex $\mathbb{R}\mathrm{P}^n$-manifolds. *American Journal of Mathematics*, 123(2):255–274, 2001.

[21] John C. Loftin. The compactification of the moduli space of convex $\mathbb{R}\mathbf{P}^2$ surfaces, I. *Journal of Differential Geometry*, 68(2):223–276, 2004.

[22] John C. Loftin. Flat metrics, cubic differentials and limits of projective holonomies. *Geometriae Dedicata*, 128(1):97–106, 2007.

[23] Xin Nie. Entropy degeneration of convex projective surfaces. *Conformal Geometry and Dynamics*, 19:318–322, 2015.

[24] Constantin Vernicos. Approximability of convex bodies and volume entropy in Hilbert geometry. *to appear in Pacific Journal of Mathematics*, 2014. arXiv:1207.1342.

[25] Scott Wolpert. A generalization of the Ahlfors-Schwarz lemma. *Proc. Amer. Math. Soc.*, 84(3):377–378, 1982.

[26] Tengren Zhang. The degeneration of convex $\mathbb{R}\mathbf{P}^2$ structures on surfaces. *Proceedings of the London Mathematical Society*, 111(5):967–1012, 2015.

University of Luxembourg, Campus Kirchberg, Mathematics Research Unit, BLG, 6, rue Richard Coudenhove-Kalergi, L-1359 Luxembourg

E-mail address: nicolas.tholozan@uni.lu